Assignment 8

Deadline: March 29 , 2019

Hand in no 1, 3, 6, 10.

Supplementary Exercises

1. Use the Weierstrass M-test to study the uniform convergence of the series $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$ for $x \in [0, b]$ where $b > 0$. The answer depends on the value of b.

2. Show that the series $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^p}$ defines a continuous function on $\mathbb{R}$ for $p > 1$.

3. Show that the infinite series $\sum_{j=1}^{\infty} \dfrac{\cos 2^j x}{3^j}$ is a continuous function on the real line. Is it differentiable?

4. Show that the sequence $g_n(x) = \sum_{j=0}^{n} e^{-jx}$ defines a smooth function on $[1, \infty)$. What will happen if $[1, \infty)$ is replaced by $[0, \infty)$?

5. Suppose f is a nonzero function satisfying $f(x + y) = f(x)f(y)$ for all real numbers x and y and is differentiable at $x = 0$. Show that it must be of the form e^{ax} for some number a. Hint: Study the differential equation f satisfies. Show that $f(0) = 1$ first.

6. (a) Show that

$$1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!} \le E(x) \le 1 + \frac{x}{1!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{e^a x^n}{n!} , \quad x \in [0, a] .$$

   (b) Show that e is not a rational number. Suggestion: Deduce from (a) the inequality

$$0 < en! - \left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) n! < \frac{e}{n+1} .$$

7. Show that the series

$$\sum_{j=0}^{\infty} \frac{x^j}{j!}$$

   is not uniformly convergent on $\mathbb{R}$ (although it is uniformly convergent in every $[-M, M]$).

8. This exercise suggests an alternative way to define the logarithmic and exponential functions. Define $\mathrm{nog} : (0, \infty) \to \mathbb{R}$ by

$$\mathrm{nog}(x) = \int_1^x \frac{1}{t} dt.$$

   (a) $\mathrm{nog}(x)$ is strictly increasing, concave, and tends to ∞ and $-\infty$ as $x \to \infty$ and 0 respectively.

   (b) $\mathrm{nog}(xy) = \mathrm{nog}(x) + \mathrm{nog}(y)$.

(c) Define $e(x)$ to be the inverse function of nog. Show that it coincides with $E(x)$.

Note: f is concave means $-f$ is convex. You cannot assume $\log x$ has been defined.

9. (a) Show that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t}dt \ .$$

Suggestion: Think about

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + \frac{(-x)^n}{1+x} \ .$$

(b) Show that

$$\left|\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| \le \frac{x^{n+1}}{n+1} \ .$$

10. (a) Show that there is a unique solution $c(x), x \in \mathbb{R}$, to the problem

$$f'' = f, \quad f(0) = 1, \ f'(0) = 0.$$

To establish uniqueness, follow what we did for the exponential function. Turn the equation into one equation with twice integration.

(b) Letting $s(x) \equiv c'(x)$, show that s satisfies the same equation as c but now $s(0) = 0, \ s'(0) = 1$.

(c) Establish the identities, for all x,

$$c^2(x) - s^2(x) = 1,$$

and

$$c(x+y) = c(x)c(y) + s(x)s(y).$$

(d) Express c and s as linear combinations of e^x and e^{-x}. (c and s are called the hyperbolic cosine and sine functions respectively. The standard notations are $\cosh x$ and $\sinh x$. Similarly one can define other hyperbolic trigonometric functions such as $\tanh x$ and $\coth x$.)